Explicit Description of a Class of Indecomposable Injective Modules*

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Abstract

Let R be a commutative Noetherian ring and $\mathfrak{p}$ be a prime ideal of R such that the ideal $\mathfrak{p}R_{\mathfrak{p}}$ is principal and $\mathrm{ht}(\mathfrak{p}) \neq 0$. In this note, we describe the explicit structure of the injective envelope of the R-module $R/\mathfrak{p}$.

Keywords: Noetherian ring, Injective module, Indecomposable injective module, Injective envelope, Weakly locally principal prime ideal.

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1 Introduction

According to classic results of E. Matlis [5, Theorems 2.5 and 2.7] every injective module over a Noetherian ring R can be expressed uniquely as the direct sum of indecomposable injective modules; the indecomposables have the form $\mathrm{E}_R(R/J)$ where J is an irreducible left ideal of R [5, Theorem 2.4]; and if in addition R is commutative the indecomposables are exactly the envelopes $\mathrm{E}_R(R/\mathfrak{p})$, $\mathfrak{p}$ being a prime ideal of R [5, Proposition 3.1]. Thus if we wish to understand the structure of the injective modules in detail, it suffices to know the structure of the indecomposables. Finding a precise description of a class of indecomposable injective modules was the main object of [7], [2], [4], [10], [9], and [1], although even over commutative Noetherian ring their structure can still be quite complicated.

In this note we give the explicit structure of the injective envelope of the R-module $R/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of R with $\mathrm{ht}(\mathfrak{p}) \neq 0$ and is *weakly locally principal*, i.e., a prime ideal of R such that there exists an element p of R for which $\mathfrak{p}R_{\mathfrak{p}} = pR_{\mathfrak{p}}$. Note that the prime ideals $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 1$ in regular rings, Krull rings and Noetherian normal rings are weakly locally principal. In particular, each prime ideal of Dedekind domain is weakly locally principal too.

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2 Main Results

Throughout this section, let R denote a commutative ring with identity, M be a unitary left R-module and $\mathrm{E}_R(-)$ denote the injective envelope of R-module $-$. Also, if $\mathfrak{p}$ denotes the weakly locally principal prime ideal of R, then p denotes the element of R for which $\mathfrak{p}R_{\mathfrak{p}} = pR_{\mathfrak{p}}$. For such $\mathfrak{p}$ and p, define $S = \{p^i s : \ s \in R \setminus \mathfrak{p}, \ i \geq 0\}$. Clearly S is a multiplicative closed subset of R and we have $R \setminus \mathfrak{p} \subseteq S$. In this case, for such S, the function $\Theta : \ R_{\mathfrak{p}} \longrightarrow S^{-1}R$ defined by $\Theta(r/s) = r/s$ is an R-homomorphism.

In the following theorem, the explicit structure of a class of indecomposable injective modules will be given.

Main Theorem *Let R be a Noetherian ring and $\mathfrak{p}$ a weakly locally principal prime ideal of R. If $\mathrm{ht}(\mathfrak{p}) \neq 0$, then $\mathrm{E}_R(R/\mathfrak{p}) \cong S^{-1}R/\Theta(R_{\mathfrak{p}})$ as R-modules.*

Let $\mathfrak{a}$ be an ideal of R. For each R-module M, set $\Gamma_{\mathfrak{a}}(M) = \bigcup_{n \in \mathbb{N}}(0 :_M \mathfrak{a}^n)$, the set of elements of M which are annihilated by some power of $\mathfrak{a}$.

For the proof of the Main Theorem we need to prove the following lemmas.

Lemma 2.1 *Let M and E be R-modules and E be injective. If a is an element of R such that $aM = M$, $\Gamma_{aR}(M) = M$, $\Gamma_{aR}(E) = E$ and $(0 :_M a) \cong (0 :_E a)$, then $M \cong E$.*

Proof. By hypothesis, there is an R-isomorphism $\varphi : \ (0 :_M a) \longrightarrow (0 :_E a)$ and therefore we obtain the induced R-monomorphism $\hat{\varphi} : \ (0 :_M a) \longrightarrow E$. Now, injectivity of E implies that there is an R-homomorphism $\psi : \ M \longrightarrow E$ such that $\psi|_{(0:_M a)} = \hat{\varphi}$. We claim that ψ is an R-isomorphism.

If K is an R-module such that $\Gamma_{aR}(K) = K$, then for $x \in K \setminus \{0\}$ we define $\exp(x) = \min\{n \in \mathbb{N} : \ a^n x = 0\}$ and we set $\exp(0) = 0$.

ψ is injective: We show that $x \in \mathrm{Ker}\ \psi$ implies $x = 0$. We use induction on $\exp(x)$. If $x \in \mathrm{Ker}\ \psi$ and $\exp(x) = 1$, then $ax = 0$, so $x \in (0 :_M a)$. Therefore $0 = \psi(x) = \hat{\varphi}(x)$ and so $x = 0$. Now suppose, inductively, $x \in \mathrm{Ker}\ \psi, \ \exp(x) = n > 1$ and suppose for each $y \in \mathrm{Ker}\ \psi$ with $\exp(y) = n - 1$, we have shown that $y = 0$. The condition $\exp(x) = n$ implies that $\exp(ax) = n - 1$. But $ax \in \mathrm{Ker}\ \psi$, so, by the inductive hypothesis, $ax = 0$. Since $n > 1$, we have $x = 0$. This completes the inductive step.

ψ is surjective: Again we use induction. Suppose $y \in E$ and $\exp(y) = 1$. Then

$ay = 0$ and we have $y \in (0 :_E a)$. Now surjectivity of φ implies that there is $x \in (0 :_M a) \subseteq M$, such that $y = \varphi(x) = \hat{\varphi}(x) = \psi(x)$, so $y \in \mathrm{Im}\ \psi$. Now suppose, inductively, $y \in E$, $\exp(y) = n > 1$ and suppose for each $z \in E$ with $\exp(z) = n-1$, we have shown that $z \in \mathrm{Im}\ \psi$. Since $y \in E$ and $\exp(y) = n$ implies that $\exp(ay) = n-1$, by the inductive hypothesis there is $x' \in M$ such that $\psi(x') = ay$. Since $aM = M$, $x' = ax'_a$, where $x'_a \in M$. Now we have $\psi(x'_a) = y$ or $\exp\big(\psi(x'_a) - y\big) = 1$. In either case we have $y \in \mathrm{Im}\ \psi$. This completes the inductive step.

Therefore we establish the claim and so $M \cong E$. $\square$

Lemma 2.2 *Let R be a Noetherian ring and $\mathfrak{p}$ be a weakly locally principal prime ideal of R for which* $\mathrm{ht}(\mathfrak{p}) \neq 0$. *Then $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong (0 :_{S^{-1}R/\Theta(R_\mathfrak{p})} p)$ as R-modules.*

Proof. Define $\phi :\ R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \longrightarrow (0 :_{S^{-1}R/\Theta(R_\mathfrak{p})} p)$ by $\phi(r/s + \mathfrak{p}R_\mathfrak{p}) = r/sp + \Theta(R_\mathfrak{p})$. Clearly ϕ is an R-homomorphism. Firstly, we prove that ϕ is surjective. For showing this, suppose $\alpha/p^it + \Theta(R_\mathfrak{p}) \in (0 :_{S^{-1}R/\Theta(R_\mathfrak{p})} p)$. Therefore, there exist $t' \in R \setminus \mathfrak{p}$ and $\beta \in R$ for which $p\alpha/p^it = \beta/t'$ or $\alpha/p^it = \beta/pt'$ in $S^{-1}R$. Now, $\phi(\beta/t' + \mathfrak{p}R_\mathfrak{p}) = \beta/pt' + \Theta(R_\mathfrak{p}) = \alpha/p^it + \Theta(R_\mathfrak{p})$ implies that ϕ is surjective. Secondly, we claim that ϕ is injective. Suppose the contrary, i.e., there is a non-zero element in $\mathrm{Ker}\ \phi$, say $r/s + \mathfrak{p}R_\mathfrak{p}$. So there exists $r'/s' \in \Theta(R_\mathfrak{p})$ such that $r/sp = r'/s'$ in $S^{-1}R$ and $r/s \notin \mathfrak{p}R_\mathfrak{p}$. Therefore there exists $l \geq 0$ and $t \in R \setminus \mathfrak{p}$ for which $p^l trs' = p^{l+1}tr's$. Consequently $(\mathfrak{p}R_\mathfrak{p})^{l+1} = (\mathfrak{p}R_\mathfrak{p})^l$. Nakayama Lemma now implies that $\mathrm{ht}(\mathfrak{p}R_\mathfrak{p}) = 0$, a contradiction. So the claim is proved and ϕ is an R-isomorphism and the lemma holds. $\square$

Proof of the Main Theorem.

Using [8, Lemma 4.24], we get $(0 :_{\mathrm{E}_R(R/\mathfrak{p})} p) \cong (0 :_{\mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})} \mathfrak{p}R_\mathfrak{p}) \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. On the other hand, Lemma 2.2 implies that $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong (0 :_{S^{-1}R/\Theta(R_\mathfrak{p})} p)$. Therefore we have

$$(0 :_{\mathrm{E}_R(R/\mathfrak{p})} p) \cong (0 :_{S^{-1}R/\Theta(R_\mathfrak{p})} p).$$

It is easy to see that $p\big(S^{-1}R/\Theta(R_\mathfrak{p})\big) = S^{-1}R/\Theta(R_\mathfrak{p})$. Now, suppose $r/p^is + \Theta(R_\mathfrak{p}) \in S^{-1}R/\Theta(R_\mathfrak{p})$. Therefore, $p^i\big(r/p^is + \Theta(R_\mathfrak{p})\big) = \Theta(R_\mathfrak{p})$ and so $r/p^is + \Theta(R_\mathfrak{p}) \in \Gamma_{pR}\big(S^{-1}R/\Theta(R_\mathfrak{p})\big)$. This shows that $\Gamma_{pR}\big(S^{-1}R/\Theta(R_\mathfrak{p})\big) = S^{-1}R/\Theta(R_\mathfrak{p})$. Since $\Gamma_{pR}\big(\mathrm{E}_R(R/\mathfrak{p})\big) = \mathrm{E}_R(R/\mathfrak{p})$ (see [6, Theorem 18.4 (v), (vi)]), Lemma 2.1 implies that

$$\mathrm{E}_R(R/\mathfrak{p}) \cong S^{-1}R/\Theta(R_\mathfrak{p}). \quad \square$$

Let p be an element of R and $\lambda :\ R \longrightarrow R_p$ be natural R-homomorphism. Then we denote the R-module $R_p/\lambda(R)=\{a/p^n + \lambda(R) :\ a \in R,\ n \geq 0\}$ by R_{p^∞}.

Proposition 2.3 *Let p be an element of R such that $\mathfrak{p} = pR$ is a maximal ideal of R. Then $\mathfrak{p}$ is the weakly locally principal prime ideal of R and if we consider $S = \{p^i s :\ s \in R \setminus \mathfrak{p},\ i \geq 0\}$ and $\Theta :\ R_{\mathfrak{p}} \longrightarrow S^{-1}R$ as we mentioned earlier we obtain $S^{-1}R/\Theta(R_{\mathfrak{p}}) \cong R_{p^\infty}$ as R-modules.*

Proof. For each $r \in R \setminus pR$ and each $l \geq 0$, there exists $\alpha,\ \beta \in R$ such that $\alpha r + \beta p^l = 1$. By using this fact, it is easy to see that the natural R-homomorphism $\mu :\ R_{p^\infty} \longrightarrow S^{-1}R/\Theta(R_{\mathfrak{p}})$ given by $\mu\big(r/p^n + \lambda(R)\big) = r/p^n + \Theta(R_{\mathfrak{p}})$ is an R-isomorphism. $\square$

Now the Main Theorem and Proposition 2.3 imply the following corollaries.

Corollary 2.4 *Let R be a Noetherian ring and $\mathfrak{p} = pR$ be a maximal ideal of R. If $\mathrm{ht}(\mathfrak{p}) \neq 0$, then $\mathrm{E}_R(R/\mathfrak{p}) \cong R_{p^\infty}$ as R-modules.*

Corollary 2.5 *Let R be a Noetherian integral domain. If pR is a non-zero maximal ideal of R, then $\mathrm{E}_R(R/pR) \cong R_{p^\infty}$ as R-modules. In particular, if p is a prime integer, then $\mathrm{E}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-modules.*

We now apply the result of Corollary 2.5 to find a decomposition for injective modules over one-dimensional unique factorization domains. In the following, $\mu(-, M)$ denotes the 0-th Bass number of M with respect to prime ideal $-$. For an R-module N, $\bigoplus \mu(-, M)N$ denotes the direct sum of $\mu(-, M)$ copies of N and consider $\Pi = \{p \in R \setminus \{0\} :\ pR \in \mathrm{Ass}_R(M)\}$, where $\mathrm{Ass}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) :$ there exists $x \in M$ such that $\mathfrak{p} = (0 :_R x)\}$.

Corollary 2.6 *Let R be a one-dimensional unique factorization domain, F its field of fractions and let M be an injective R-module. Then*

$$M \cong \big(\bigoplus \mu(0, M)F\big) \oplus \big(\bigoplus_{p \in \Pi} \mu(pR, M)R_{p^\infty}\big)$$

as R-modules.

We need the following lemma to prove this corollary.

Lemma 2.7 *The ring R is a principal ideal domain if and only if R is a one-dimensional unique factorization domain.*

Proof. Clearly any principal ideal domain is a unique factorization domain and one-dimensional, so we prove the converse which is more interesting. Suppose that R is a one-dimensional unique factorization domain. We note that the proof of Theorem 20.1 in [6] shows that if R is a unique factorization domain and $\mathfrak{p}$ is a prime ideal of R such that $\mathrm{ht}(\mathfrak{p}) = 1$, then $\mathfrak{p}$ is a principal ideal. Since R is one-dimensional, any prime ideal of R is principal. So R is Noetherian. Now if R is not a principal ideal domain, there is a non-principal ideal $\mathfrak{a}$. Since R is Noetherian, there is an ideal, $\mathfrak{m}$, that is maximal with respect to being non-principal. A standard result of M. Isaacs (see [3, page 8]) states that $\mathfrak{m}$ is a prime ideal. This contradicts the fact that all prime ideals of R are principal and completes the proof. $\square$

Proof of the Corollary 2.6.

Let $0 \neq \mathfrak{p} \in \mathrm{Ass}_R(M)$. By Lemma 2.7, $\mathfrak{p} = pR$ for some $p \in \Pi$. We know that M has a decomposition in the form

$$M \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} \mu(\mathfrak{p}, M)\mathrm{E}_R(R/\mathfrak{p})$$

(see [5, Theorem 2.5 and Proposition 3.1]). But it is easy to see that $\mu(\mathfrak{p}, M) \neq 0$ if and only if $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Therefore since $\mathrm{E}_R(R) \cong F$ we have

$$\begin{aligned} M &\cong \textstyle\bigoplus_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mu(\mathfrak{p}, M)\mathrm{E}_R(R/\mathfrak{p}) \\ &\cong \big(\textstyle\bigoplus \mu(0, M)\mathrm{E}_R(R)\big) \oplus \big(\textstyle\bigoplus_{0 \neq \mathfrak{p} \in \mathrm{Ass}_R(M)} \mu(\mathfrak{p}, M)\mathrm{E}_R(R/\mathfrak{p})\big) \\ &\cong \big(\textstyle\bigoplus \mu(0, M)F\big) \oplus \big(\textstyle\bigoplus_{p \in \Pi} \mu(pR, M)R_{p^\infty}\big). \quad \square \end{aligned}$$

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